AN INFINITESIMAL-BIRATIONAL DUALITY THROUGH DIFFERENTIAL OPERATORS

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Abstract. The structure of filtered algebras of Grothendieck's differential operators of truncated polynomials in one variable and graded Poisson algebras of their principal symbols is explicitly determined. A related infinitesimal-birational duality realized by a Springer type resolution of singularities and the Fourier transformation is presented. This algebro-geometrical duality is quantized in appropriate sense and its quantum origin is explained.

It seems that our ship is arriving at a coast,
but the land is still hidden in the fog.

1. Introduction. The study of the algebra of differential operators of a commutative algebra was initiated by Grothendieck [19] in 1967. The most important question in this theory is how properties of differential operators reflect the properties of a commutative algebra. Grothendieck proved that the algebra of differential operators on a smooth affine variety over a field of characteristic zero is generated by operators of order at most one [19], in particular is finitely generated. The converse, that generation by first order filtration implies smoothness, is now referred to as Nakai's conjecture. It implies the Zariski-Lipman conjecture [34], [3], [22]. Nakai's conjecture and its variants has been verified only for few classes of commutative algebras in [41], [24], [25], [42], [52], [35], [50], [49]. In general, the determination of the structure of differential operators of a given non-smooth commutative algebra is a hard problem and one expects some pathology caused by singularities. For example, the algebra of differential operators on the cubic cone $x^3 + y^3 + z^3 = 0$ is not generated by operators of bounded order, in particular is not finitely generated [5]. Although for reduced curves the algebra of differential operators is Noetherian [36], [44], for the non-reduced commutative algebra of Krull dimension one, $k[x, y]/(x^2, xy)$, its algebra of differential operators is right but not left Noetherian [36]. Such and other problems concerning the structure of the algebra of differential operators can be found also in [7], [8], [9], [10], [27], [32], [33], [36], [37], [38], [39], [40], [43], [44], [48], [51].

In this paper we determine the structure of the filtered algebra $\mathcal{D}(\mathcal{O}_n)$ of differential operators of the polynomial algebra $\mathcal{O}_n = k[x]/(x^{n+1})$ of the n-th infinitesimal neighborhood of a point on the affine line and its associated graded Poisson algebra $\mathcal{P}(\mathcal{O}_n)$. It turns out that instead of expected pathology we find beautiful and intriguing phenomena.

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Let (e,h,f) be the standard basis of sl_2, subject to the relations: $[e,f]=h,\ [h,e]=2e,\ [h,f]=-2f$. We show that there exists an extension of filtered algebras

$$(e^{n+1})\rightarrowtail \mathcal{U}(\mathrm{sl}_2)/(C-n(n+2))\twoheadrightarrow \mathcal{D}(\mathcal{O}_n),$$

where $C=h^2+2(ef+fe)$ is the Casimir element and we filter the enveloping algebra giving (e,h,f) orders $(0,1,2)$. The epimorphism maps (e,h,f) onto operators

$$(x,\quad 2x\frac{\mathrm{d}}{\mathrm{d}x}-n,\quad -x\frac{\mathrm{d}^2}{\mathrm{d}x^2}+n\frac{\mathrm{d}}{\mathrm{d}x}).$$

In particular, $\mathcal{D}(\mathcal{O}_n)$ for every n is generated by operators of order less or equal 2, or more precisely, by x and a single operator of order 2. This means that algebras $\mathcal{O}_n$ of n-th infinitesimal neighborhoods of a one-dimensional smooth point are close to smooth algebras.

After passing to the associated graded Poisson algebras we get the extension

$$\mathfrak{m}^{n+1}\rightarrowtail \mathcal{P}\twoheadrightarrow \mathcal{P}(\mathcal{O}_n),$$

where $\mathcal{P}$ is the algebra of polynomial functions on the nilpotent cone in sl_2^* with its canonical graded Kirillov-Kostant Poisson structure and $\mathfrak{m}$ is the maximal ideal of the vertex. Both extensions are sl_2-invariant in an appropriate way. In the second extension the epimorphism describes the closed embedding via the moment map associated with a hamiltonian action of sl_2 on $\mathrm{Spec}(\mathcal{P}(\mathcal{O}_n))$.

Note that the first extension can be regarded as a quantization of the second extension of their associated commutative graded Poisson algebras. Although maximal primitive quotients of the enveloping algebra $\mathcal{U}(\mathrm{sl}_2)/(C-n(n+2))$ depend on n, they are Morita equivalent one to each other by the Beilinson-Bernstein theorem [4]. This means that the Morita equivalence class of $\mathcal{U}(\mathrm{sl}_2)/(C-n(n+2))$, a quantization of the nilpotent cone in sl_2^*, does not depend on n. It is easy to see that forgetting about filtration one gets $\mathcal{D}(\mathcal{O}_n)\cong \mathrm{End}_k(\mathcal{O}_n)$ so for all n algebras $\mathcal{D}(\mathcal{O}_n)$ are Morita equivalent. This means that for all n the above quantizations of n-th infinitesimal neighborhoods of the vertex of the nilpotent cone are equivalent.

On the other hand, although for $n>0$ surjective homomorphisms of commutative algebras $\mathcal{O}_{n+1}\to\mathcal{O}_n$ do not induce homomorphisms of algebras of differential operators $\mathcal{D}(\mathcal{O}_{n+1})\to\mathcal{D}(\mathcal{O}_n)$ (there is no any homomorphism of algebras of matrices $\mathcal{M}_{n+1}\to\mathcal{M}_n$), we have surjective homomorphisms of their graded Poisson algebras $\mathcal{P}(\mathcal{O}_{n+1})\to\mathcal{P}(\mathcal{O}_n)$ which is a kind of surprise, which cannot be explained by classical geometry.

Next, after passing to the inverse limits, we compare the completion of the algebra of principal symbols on the affine line with the inverse limit of the above system of algebras of principal symbols on infinitesimal neighborhoods of a point in the affine line. We show that there exists a unique grading preserving homomorphism of graded Poisson algebras over the completed local ring of a point in the line, between these two limits. Geometrically, this is a completion of a resolution of singularities in the category of conical Poisson varieties

$$\begin{tikzcd} \mathrm{T}^*\mathbb{A}^1 \arrow[rr, "\exists!"] \arrow[dr] & & Y \arrow[dl] \\ & \mathbb{A}^1 & \end{tikzcd}.$$

Finally, we show that the above canonical resolution of singularity together with the Fourier transformation establish a **duality** between the above SL_2-symmetries of principal symbols of the system of infinitesimal neighborhoods of finite order of a point in the line $\mathbb{A}^1$ (**infinitesimal** picture) and automorphisms of dual principal symbols induced by birational automorphisms of the dual line $\hat{\mathbb{A}}^1$ (**birational** picture).

"Magic Diamond Theorem". *The following diagram of canonical morphisms in the category of conical Poisson varieties*

$$\begin{tikzcd}
 & \mathrm{T}^*\mathbb{A}^1 \arrow[dl, "\text{resol. sing.}"'] \arrow[dr, "\text{cl. Fourier tr.}"] & \\
Y \arrow[dr, "\text{inf. moment m.}"'] & & \mathrm{T}^*\hat{\mathbb{A}}^1 \arrow[dl, "\text{bir. moment m.}"] \\
 & \mathrm{sl}_2^* &
\end{tikzcd}$$

is commutative.

This diagram can be quantized in appropriate sense, which recovers quantum nature of classical algebraic geometry as follows

"Quantized Magic Diamond Theorem". *There exists the following commutative diagram in the category of almost commutative algebras*

$$\begin{tikzcd}
 & \mathcal{D}(\mathbb{A}^1) & \\
\mathcal{U}(\mathrm{sl}_2)/(C-n(n+2)) \arrow[ur, "\text{q. resol. sing.}"] & & \mathcal{D}(\hat{\mathbb{A}}^1) \arrow[ul, "\text{q. Fourier tr.}"'] \\
 & \mathcal{U}(\mathrm{sl}_2) \arrow[ul, "\text{q. inf. moment m.}"] \arrow[ur, "\text{q. bir. moment m.}"'] &
\end{tikzcd}$$

which is a quantization of the Magic Diamond.

It should be understood as a purely mathematical analog of the physical de Broglie particle-wave duality [16], where an infinitesimal neighbourhood of a closed point on the line and the general point of the dual line are mathematical counterparts of a spatially localized particle and its Fourier decomposition into waves of frequencies spread in continuous spectrum, respectively. The de Broglie particle-wave duality could not be explained in terms of classical physics and was the main motivation for the new quantum physics. Our infinitesimal-birational duality cannot be explained in terms of classical geometry (what we emphasize in the name of the above theorem). In both dualities the non-classical part is the Fourier transformation realizing both dualities in a similar way. This suggests that classical geometry should be "quantized" in an appropriate sense in order to explain the perfect harmony of the following aspects of our example:

- $\mathcal{D}$-modules on non-reduced schemes.
- Representation theory and moment maps.
- Resolution of singularities.
- Fourier transformation.

It should be noted that our result is not an isolated "quantum" phenomenon relating algebraic geometry, representation theory, moment maps and the Fourier transformation. One attempt in this direction is the general program of deformation quantization of nilpotent coadjoint orbits of complex semisimple Lie

groups [1], [2], [12], [14], [17], and deformation quantization of moment maps [20], [21], [55]. Another program concentrates around Vogan's conjecture relating the Fourier transform of a nilpotent orbit of the coadjoint representation of a complex Lie group and multiplicities of the ring of polynomial functions on that orbit [29], [30], [31], [54]. Moreover, the Fourier transformation of sheaves supported on the nilpotent cone in the co-adjoint representation of a reductive algebraic group play an important role in the theory of Springer representations of the Weyl group on intersection cohomology [6], [11], [18], [23], [28], [45], [46], [47].

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2. Basic definitions. Let k be a field of characteristic zero.

Definition 1. *A filtered associative algebra $\mathcal{D} = \bigcup_{p\geq 0} \mathcal{D}^p$, $\mathcal{D}^p \subset \mathcal{D}^{p+1}$, is called **almost commutative** if $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q-1}$.*

Definition 2. *Given a commutative algebra $\mathcal{O}$ one defines the filtered **algebra** $\mathcal{D}(\mathcal{O}) = \bigcup_{p\geq 0} \mathcal{D}^p(\mathcal{O})$ **of differential operators of** $\mathcal{O}$, where*

$$\mathcal{D}^p(\mathcal{O}) := \{\delta \in \mathrm{End}_k(\mathcal{O}) \mid \forall_{f_0,\ldots,f_p \in \mathcal{O}} \quad [f_0, \ldots, [f_p, \delta]\ldots] = 0\}.$$

*This filtration is called **order filtration**.*

One shows that the filtered associative algebra $\mathcal{D}(\mathcal{O})$ is almost commutative.

Definition 3. *A graded commutative algebra $\mathcal{P} = \bigoplus_{p\geq 0} \mathcal{P}^p$ is called **graded Poisson algebra** if there is given a Lie algebra structure on $\mathcal{P}$*

$$\{-,-\} : \mathcal{P} \otimes \mathcal{P} \to \mathcal{P},$$

such that

$$\{\mathcal{P}^p, \mathcal{P}^q\} \subset \mathcal{P}^{p+q-1}$$

and for all $f, g, h \in \mathcal{P}$

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Example. The associated graded commutative algebra $\mathrm{Gr}(\mathcal{D}) := \bigoplus_{p\geq 0} \mathcal{D}^p/\mathcal{D}^{p-1}$ of an almost commutative filtered algebra $\mathcal{D}$ is a graded Poisson algebra, where for all $\delta \in \mathcal{D}^p$, $\epsilon \in \mathcal{D}^q$

$$\{\delta + \mathcal{D}^{p-1}, \epsilon + \mathcal{D}^{q-1}\} := [\delta, \epsilon] + \mathcal{D}^{p+q-2}.$$

Definition 4. *Homogeneous elements of a graded Poisson algebra $\mathcal{P}(\mathcal{O}) := \mathrm{Gr}(\mathcal{D}(\mathcal{O}))$ are called **principal symbols of differential operators of** $\mathcal{O}$.*

For $X = \mathrm{Spec}(\mathcal{O})$ we will write $\mathcal{D}(X) := \mathcal{D}(\mathcal{O})$, $\mathcal{P}(X) := \mathcal{P}(\mathcal{O})$.

Example. Let $\mathfrak{g} = \bigcup_{p\geq 0} \mathfrak{g}^p$, $\mathfrak{g}^p \subset \mathfrak{g}^{p+1}$, be a filtered Lie algebra, i.e.

$$[\mathfrak{g}^p, \mathfrak{g}^q] \subset \mathfrak{g}^{p+q-1}.$$

Let us give its enveloping algebra $\mathcal{U}(\mathfrak{g})$ the induced filtration. Then $\mathcal{U}(\mathfrak{g})$ becomes an almost commutative algebra whose associated graded Poisson algebra

$\mathrm{Gr}(\mathcal{U}(\mathfrak{g}))$ is the symmetric algebra $\mathrm{S}(\mathrm{Gr}(\mathfrak{g}))$ of the associated graded vector space $\mathrm{Gr}(\mathfrak{g}) = \bigoplus_{\mathrm{p}\geq 0} \mathfrak{g}^{\mathrm{p}}/\mathfrak{g}^{\mathrm{p}-1}$, where for generators $X \in \mathfrak{g}^p, Y \in \mathfrak{g}^q$

$$\{X + \mathfrak{g}^{p-1}, Y + \mathfrak{g}^{q-1}\} := [X, Y] + \mathfrak{g}^{p+q-2}.$$

Any ad-invariant homogeneous ideal in $\mathrm{S}(\mathrm{Gr}(\mathfrak{g}))$ defines a graded Poisson structure on the factor-algebra. One calls it **graded Kirillov-Kostant Poisson structure**.

Definition 5. *Let $\mathcal{P} = \bigoplus_{p\geq 0} \mathcal{P}^p$ be a graded Poisson algebra. A **quantization** of $\mathcal{P}$ is a pair consisting of the Morita equivalence class of an almost commutative algebra $\mathcal{D} = \bigcup_{p\geq 0} \mathcal{D}^p$ together with an isomorphism of graded Poisson algebras $\mathrm{Gr}(\mathcal{D}) \to \mathcal{P}$.*

Note that any almost commutative algebra is a quantization of its associated graded Poisson algebra.

Definition 6. *A morphism of conical Poisson schemes*

$$\mathrm{Spec}(\mathcal{P}) \to \mathrm{Spec}(\mathrm{S}(\mathrm{Gr}(\mathfrak{g})))$$

determined by a homomorphism of graded Poisson algebras

$$\mathrm{S}(\mathrm{Gr}(\mathfrak{g})) \to \mathcal{P}$$

*is called **moment map**.*

Definition 7. *For a given morphism of conical Poisson schemes $\mathrm{Spec}(\mathcal{P}_1) \to \mathrm{Spec}(\mathcal{P}_2)$ its **quantization** is a pair consisting of the Morita equivalence class of a filtration preserving homomorphism of almost commutative algebras*

$$\mathcal{D}_2 \to \mathcal{D}_1$$

and an isomorphism of its associated grading preserving homomorphism of graded Poisson algebras with a given homomorphism of graded Poisson algebras $\mathcal{P}_2 \to \mathcal{P}_1$. The definition of a quantization of a given diagram in the category of conical Poisson schemes is obvious.

Note that the set of k-points of $\mathrm{Spec}(\mathrm{S}(\mathrm{Gr}(\mathfrak{g})))$ can be canonically identified with the dual space $\mathrm{Gr}(\mathfrak{g})^*$. Since all our structures are defined over k we will abuse the language and will not distinguish $\mathrm{Spec}(\mathrm{S}(\mathrm{Gr}(\mathfrak{g})))$ from $\mathrm{Gr}(\mathfrak{g})^*$. In the case of a filtered Lie algebra coming from a graded Lie algebra $\mathfrak{g} = \bigoplus_{i\geq 0} \mathfrak{g}_i$, $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j-1}$, i.e. when the filtration has the form $\mathfrak{g}^p = \bigoplus_{0\leq i\leq p} \mathfrak{g}_i$, we will not distinguish also the graded vector space $\mathrm{Gr}(\mathfrak{g})$ from the graded Lie algebra $\mathfrak{g}$.

If one gives a commutative algebra $\mathcal{O}$ the trivial grading and the trivial Poisson structure then one has the canonical injective homomorphism of graded Poisson algebras $\mathcal{O} \rightarrowtail \mathcal{P}(\mathcal{O})$ onto the subalgebra of symbols of degree zero, and the canonical surjective homomorphism of graded $\mathcal{O}$-algebras $\mathcal{P}(\mathcal{O}) \twoheadrightarrow \mathcal{O}$ annihilating all symbols of positive degree. If $\mathcal{O} = \mathcal{O}(V)$ is the algebra of polynomial functions on a smooth affine variety V then $\mathrm{Spec}(\mathcal{P}(\mathcal{O})) = \mathrm{T}^*V$, the cotangent bundle of V. Then the above homomorphisms describe the canonical projection $\mathrm{T}^*V \to V$ and the embedding onto the zero section $V \to \mathrm{T}^*V$, respectively.

Let E be a finite dimensional graded vector space and $\hat{E} = E^*[1]$ be its dual with the dual grading enlarged by one. Then

$$\mathrm{T}^*\mathrm{Spec}(\mathrm{Sym}(E)) = \mathrm{Spec}(\mathrm{Sym}(E) \otimes \mathrm{Sym}(\hat{E})),$$

is a conical Poisson variety with the following Poisson structure

$$\{e_1 \otimes 1, e_2 \otimes 1\} = 0, \quad \{1 \otimes \hat{e}_1, 1 \otimes \hat{e}_2\} = 0, \quad \{1 \otimes \hat{e}, e \otimes 1\} = \hat{e}(e).$$

Using the canonical linear isomorphism of graded vector spaces $E \to \hat{\hat{E}}$, we obtain the canonical identification

$$\mathrm{T}^*\mathrm{Spec}(\mathrm{Sym}(\hat{E})) \xrightarrow{\cong} \mathrm{Spec}(\mathrm{Sym}(\hat{E}) \otimes \mathrm{Sym}(E)).$$

Definition 8. *We define an isomorphism of conical Poisson varieties*

$$\mathrm{T}^*\mathrm{Spec}(\mathrm{Sym}(E)) \to \mathrm{T}^*\mathrm{Spec}(\mathrm{Sym}(\hat{E}))$$

by means of polynomial functions as follows

$$\mathrm{Sym}(E) \otimes \mathrm{Sym}(\hat{E}) \leftarrow \mathrm{Sym}(\hat{E}) \otimes \mathrm{Sym}(E),$$

$$\hat{e} \otimes 1 \mapsto 1 \otimes \hat{e}, \quad 1 \otimes e \mapsto -e \otimes 1.$$

We call this isomorphism ***classical Fourier transformation****.*

Note that our classical Fourier transformation on the level of cotangent bundles $\mathrm{T}^*(\mathbb{A}^n) \to \mathrm{T}^*(\hat{\mathbb{A}}^n)$ can be regarded as a result of descending of the following **quantum Fourier transformation**

$$\mathcal{D}(\hat{\mathbb{A}}^n) \to \mathcal{D}(\mathbb{A}^n)$$

$$\hat{x}_i \mapsto \frac{\partial}{\partial x_i}, \quad \frac{\partial}{\partial \hat{x}_i} \mapsto -x_i$$

to the level of principal symbols, what justifies our terminology.

According to Definition 7 the quantum Fourier transformation is a quantization of the classical Fourier transformation, provided $\mathcal{D}(\hat{\mathbb{A}}^n)$ is filtered not by the order of differential operators but by the degree of their polynomial coefficients. Whenever $\mathcal{D}(\hat{\mathbb{A}}^n)$ appears in this article it is equipped with this filtration making it an almost commutative algebra.

For the affine line $\mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$ we can define the dual line as $\hat{\mathbb{A}}^1 = \hat{\mathbb{P}}^1 \setminus \{\infty\} = \mathbb{P}^1 \setminus \{0\}$, by self-duality of the projective line, and we regard $\mathrm{T}^*\hat{\mathbb{A}}^1$ as an open subscheme in $\mathrm{T}^*\hat{\mathbb{P}}^1$. The affine algebraic group SL_2 acts on $\hat{\mathbb{P}}^1$ and this action lifts canonically to the conical structure preserving Poisson action on the conical Poisson variety $\mathrm{T}^*\hat{\mathbb{P}}^1$. We can regard the restriction of this action to $\mathrm{T}^*\hat{\mathbb{A}}^1$ as the induced rational action of the pseudogroup of birational automorphisms of $\hat{\mathbb{A}}^1$. The Lie algebra sl_2 of vector fields on $\hat{\mathbb{P}}^1$ lifts canonically to $\mathrm{T}^*\hat{\mathbb{P}}^1$ and defines a moment map $\mathrm{T}^*\hat{\mathbb{P}}^1 \to \mathrm{sl}_2^*$. The restriction of this moment map to $\mathrm{T}^*\hat{\mathbb{A}}^1$ can be understood as the moment map

$$\mathrm{T}^*\hat{\mathbb{A}}^1 \to \mathrm{sl}_2^*$$

defined by the induced rational action of the pseudogroup of birational automorphisms of $\hat{\mathbb{A}}^1$.

3. Differential operators of truncated polynomials. We will study the polynomial algebra $\mathcal{O}_n := k[x]/(x^{n+1})$ of the n-th infinitesimal neighborhood of a point on the affine line, the algebra $\mathcal{D}(\mathcal{O}_n)$ of differential operators on this neighborhood and the corresponding associated graded Poisson algebra $\mathcal{P}(\mathcal{O}_n)$ spanned by their principal symbols. We identify functions from $\mathcal{O}_n$ with operators of zero order from $\mathcal{D}^0(\mathcal{O}_n)$ and their principal symbols from $\mathcal{P}^0(\mathcal{O}_n)$, for instance we identify $x^l \in \mathcal{O}_n$ with the operator $\delta_0^l \in \mathcal{D}^0(\mathcal{O}_n)$ of multiplication by x^l and with its principal symbol $y_0^l \in \mathcal{P}^0(\mathcal{O}_n)$. For any $\delta \in \mathcal{D}(\mathcal{O}_n)$ and $f \in \mathcal{O}_n$ we use the notation

$$\mathrm{ad}_{f(x)}(\delta) := [f(\delta_0), \delta].$$

Lemma 1.

$$\mathrm{ad}_x^{p+1}(\delta) = 0 \Leftrightarrow \forall_{f_0,\ldots,f_p \in \mathcal{O}_n} \quad \mathrm{ad}_{f_0}\ldots\mathrm{ad}_{f_p}(\delta) = 0.$$

Proof. One can assume $f_0 = x^{l_0}, \ldots, f_p = x^{l_p}$, for $l_0, \ldots, l_p \geq 0$. Since $\mathrm{ad}_{x^l}(\delta) = \sum_{m=1}^{l} \delta_0^{m-1}\mathrm{ad}_x(\delta)\delta_0^{l-m}$ and ad_x commutes with left and right multiplication by δ_0 we get

$$\mathrm{ad}_{x^{l_0}}\ldots\mathrm{ad}_{x^{l_p}}(\delta) =$$

$$= \sum_{v_0=1}^{l_0} \delta_0^{v_0-1}\mathrm{ad}_x(\sum_{v_1=1}^{l_1} \delta_0^{v_1-1}\mathrm{ad}_x(\ldots\delta_0^{v_p-1}\mathrm{ad}_x(\delta)\delta_0^{l_p-v_p})\ldots)\delta_0^{l_0-v_0} =$$

$$= \sum_{v_0=1}^{l_0} \ldots \sum_{v_p=1}^{l_p} \delta_0^{(v_0-1)+\ldots+(v_p-1)}\mathrm{ad}_x^{p+1}(\delta)\delta_0^{(l_0-v_0)+\ldots+(l_p-v_p)}.$$

which implies the lemma. □

Lemma 2.

$$x^{n+1} = 0 \Rightarrow \mathrm{ad}_x^{2n+1} = 0.$$

Proof. Substituting $l = 2n + 1$ into the identity

$$\mathrm{ad}_x^l(\delta) = \sum_{m=0}^{p} (-1)^{m+1} \begin{pmatrix} l \\ m \end{pmatrix} \delta_0^{l-m}\delta\delta_0^m$$

one can see that at least one of $m, l - m$ is greater than n. □

Corollary 1.

$$\mathcal{D}(\mathcal{O}_n) = \mathrm{End}_k(\mathcal{O}_n).$$

Corollary 2. *The order filtration on differential operators has the form*

$$\mathcal{D}^p(\mathcal{O}_n) = \big\{\delta \in \mathrm{End}_k(\mathcal{O}_n) \mid \mathrm{ad}_x^{p+1}(\delta) = 0\big\}.$$

Lemma 3. *We have the canonical linear embedding $\mathcal{P}^p(\mathcal{O}_n) \rightarrowtail \mathcal{O}_n$ induced by the map $\mathrm{ad}_x^p : \mathcal{D}^p(\mathcal{O}_n) \to \mathcal{O}_n$. It is multiplicative in the following sense: for all $\delta \in \mathcal{D}^p(\mathcal{O}_n)$, $\epsilon \in \mathcal{D}^q(\mathcal{O}_n)$*

$$\frac{1}{(p+q)!}\mathrm{ad}_x^{p+q}(\delta\epsilon) = \frac{1}{p!}\mathrm{ad}_x^p(\delta)\frac{1}{q!}\mathrm{ad}_x^q(\epsilon). \tag{1}$$

Proof. Since $\mathcal{D}^p(\mathcal{O}_n) = \ker \mathrm{ad}_x^{p+1}$ then ad_x^p maps $\mathcal{D}^p(\mathcal{O}_n)$ into $\mathcal{O}_n = \ker \mathrm{ad}_x$ with kernel $\mathcal{D}^{p-1}(\mathcal{O}_n) \subset \mathcal{D}^p(\mathcal{O}_n)$, hence embeds $\mathcal{D}^p(\mathcal{O}_n) = \mathcal{D}^p(\mathcal{O}_n)/\mathcal{D}^{p-1}(\mathcal{O}_n)$ into $\mathcal{O}_n$. Using the fact that $\mathrm{ad}_x^l(\delta) = 0$ and $\mathrm{ad}_x^m(\delta') = 0$ for $l > p$ and $m > q$ in the identity

$$\frac{1}{(p+q)!}\mathrm{ad}_x^{p+q}(\delta\epsilon) = \sum_{l+m=p+q} \frac{1}{l!}\mathrm{ad}_x^l(\delta)\frac{1}{m!}\mathrm{ad}_x^m(\epsilon)$$

we get on the right hand side only one summand with $l = p$ and $m = q$. $\square$

Note that $\mathrm{ad}_x^p : \mathcal{D}^p(\mathcal{O}_n) \to \mathcal{O}_n$ is $\mathcal{O}_n$-linear, hence its image is an ideal, necessarily generated by x^{v_p} for some $v_p \leq n$.

Definition 9. *We define operators $\delta_p \in \mathcal{D}^p(\mathcal{O}_n)$ for $p = 1, ..., 2n$ such that*

$$\frac{1}{p!}\mathrm{ad}_x^p(\delta_p) = \delta_0^{v_p}. \tag{2}$$

Note that operators δ_p are determined up to $\mathcal{D}^{p-1}(\mathcal{O}_n) = \ker(\mathrm{ad}_x^p)$. However, by Lemma 3, their principal symbols $y_p \in \mathcal{P}^p(\mathcal{O}_n)$ are defined uniquely.

Lemma 4. *The system $(y_0^l y_p)_{l=0}^{n-v_p}$ is a basis of $\mathcal{P}^p(\mathcal{O}_n)$ for $i = 1, ..., 2n$.*

Proof. Since on the right hand side of the equality

$$\frac{1}{p!}\mathrm{ad}_x^p(\delta_0^l\delta_p) = \delta_0^{l+v_p}$$

we get linearly independent powers of δ_0 then the system of corresponding principal symbols is linearly independent in $\mathcal{P}^p(\mathcal{O}_n)$. On the other hand for every $\delta \in \mathcal{D}^p(\mathcal{O}_n)$ we have the following decomposition

$$\mathrm{ad}_x^p(\delta) = \sum_{l=0}^{n-v_p} c_l\delta_0^{l+v_p} = \sum_{l=0}^{n-v_p} c_l\delta_0^l\delta_0^{v_p} =$$

$$= \sum_{l=0}^{n-v_p} c_l\delta_0^l\frac{1}{p!}\mathrm{ad}_x^p(\delta_p) = \mathrm{ad}_x^p(\sum_{l=0}^{n-v_p}\frac{c_l}{p!}\delta_0^l\delta_p)$$

which means that

$$\delta \equiv \sum_{l=0}^{n-v_p}\frac{c_l}{p!}\delta_0^l\delta_p \mod \mathcal{D}^{p-1}(\mathcal{O}_n).$$

Therefore the above system of principal symbols generates $\mathcal{P}^p(\mathcal{O}_n)$. $\square$

Corollary 3.

$$\dim \mathcal{P}^p(\mathcal{O}_n) = n - v_p + 1.$$

Corollary 4.

$$v_1 + ... + v_{2n} = n(n+1).$$

Proof. We have

$$(n+1)^2 = \dim \mathrm{End}_k(\mathcal{O}_n) = \dim \mathcal{D}(\mathcal{O}_n) =$$

$$= \dim \mathcal{O}_n + \sum_{p=1}^{2n} \dim \mathcal{P}^p(\mathcal{O}_n) = (n+1) + 2n(n+1) - (v_1 + ... + v_{2n}).\square$$

Our next task is to determine v_p's for $i = 1, ..., 2n$.

Lemma 5. *$v_p > 0$.*

Proof. In the ordered basis $(1, x, ..., x^n)$ of $\mathcal{O}_n$ the multiplication by x has the form of the sub-diagonal Jordan block. Therefore we have

$$\mathrm{tr}(\delta_0^{v_p}) = \frac{1}{p!}\mathrm{tr}(\mathrm{ad}_x^p(\delta_p)) = 0.$$

But it is possible only if $v_p > 0$. □

Lemma 6. *If $n > 0$ then $v_1 = v_2 = 1$.*

Proof. We know already that $v_1, v_2 \geq 1$. We show that $v_1, v_2 \leq 1$. Using the Jordan form we can easily find a solution $\delta_2 \in \mathrm{End}_k(\mathcal{O}_n)$ to the equation

$$\frac{1}{2}\mathrm{ad}_x^2(\delta_2) = \delta_0. \tag{3}$$

Therefore $v_2 = 1$. If we take $\delta_1 := \frac{1}{2}\mathrm{ad}_x(\delta_2)$ then

$$\mathrm{ad}_x(\delta_1) = \frac{1}{2}\mathrm{ad}_x^2(\delta_2) = \delta_0. \tag{4}$$

Therefore $v_1 = 1$. □

One can easily check that one can choose the operator δ_2 of the form $x\frac{\mathrm{d}^2}{\mathrm{d}x^2} - n\frac{\mathrm{d}}{\mathrm{d}x}$. Then the operators

$$\delta_0 := x, \quad \delta_1 := -x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{n}{2}, \quad \delta_2 := x\frac{\mathrm{d}^2}{\mathrm{d}x^2} - n\frac{\mathrm{d}}{\mathrm{d}x} \tag{5}$$

satisfy the relations

$$[\delta_0, \delta_1] = \delta_0, \quad [\delta_0, \delta_2] = 2\delta_1, \quad [\delta_1, \delta_2] = \delta_2, \tag{6}$$

and

$$\delta_1^2 - \frac{1}{2}(\delta_0\delta_2 + \delta_2\delta_0) = \frac{n}{2}(\frac{n}{2} + 1), \tag{7}$$

which means that we have a homomorphism of filtered algebras from a maximal primitive quotient of the enveloping algebra of sl_2 [15] into $\mathcal{D}(\mathcal{O}_n)$

$$\mathcal{U}(\mathrm{sl}_2)/(C - n(n+2)) \to \mathcal{D}(\mathcal{O}_n),$$

$$e \mapsto \delta_0, \quad h \mapsto -2\delta_1, \quad f \mapsto -\delta_2, \tag{8}$$

where sl_2 is spanned by (e, h, f) subject to the relations

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f,$$

and C denotes the Casimir element $h^2 + 2(ef + fe)$.

Definition 10. *We define on the maximal primitive quotient a filtration coming from the filtration on the Lie algebra* sl_2*: $(e) \subset (e, h) \subset (e, h, f) = \mathrm{sl}_2$. We call the above choice* (5) *of $\delta_0, \delta_1, \delta_2$ and the resulting homomorphism* (8) *of filtered algebras* ***distinguished****.*

Note that this filtration differs from the standard filtration of the enveloping algebra of a Lie algebra, under which the Lie algebra entirely lies in the first piece of the filtration.

Lemma 7. *If $n > 0$ then $v_{2l-1} = v_{2l} = l$ for $l = 1, ..., n$.*

Proof. Note first that if $x^l \in \mathrm{ad}_x^p(\mathcal{D}(\mathcal{O}_n)^p) \subset \mathcal{O}_n$ then $v_p \leq l$. Using the multiplicative law from Lemma 3 and the equalities $v_1 = v_2 = 1$ we get

$$\frac{1}{(2l)!}\mathrm{ad}_x^{2l}(\delta_2^l) = (\frac{1}{2}\mathrm{ad}_x^2(\delta_2))^l = \delta_0^l, \tag{9}$$

$$\frac{1}{(2l-1)!}\mathrm{ad}_x^{2l-1}(\delta_1\delta_2^{l-1}) = \mathrm{ad}_x(\delta_1)(\frac{1}{2}\mathrm{ad}_x^2(\delta_2))^{l-1} = \delta_0^l, \tag{10}$$

which means that $v_{2l-1}, v_{2l} \leq l$. But on the other hand $v_{2l-1}, v_{2l} \geq 0$ and

$$n(n+1) = v_1 + ... + v_{2n} = \sum_{l=1}^{n}(v_{2l-1} + v_{2l}) \leq \sum_{l=1}^{n}(l+l) = n(n+1),$$

which implies the desired equalities. □

Definition 11. *We will identify graded algebras* $\mathrm{Sym}(\mathrm{sl}_2)$ *and* $k[z_0, z_1, z_2]$ *using*

$$z_0 = e, \quad z_1 = -\frac{1}{2}h, \quad z_2 = -f,$$

where deg $z_p = p$*. Then the canonical Kirillov-Kostant Poisson structure takes the following form*

$$\{z_0, z_1\} = z_0, \quad \{z_0, z_2\} = 2z_1, \quad \{z_1, z_2\} = z_2.$$

Let $\mathcal{P} := k[z_0, z_1, z_2]/(z_1^2 - z_0z_2)$ *be a graded polynomial algebra of the Zariski closure of the unique not closed* $\mathrm{Ad}^*_{\mathrm{SL}_2}$*-orbit in* sl_2^* *(which here can be identified via the Killing form with the nilpotent cone in* sl_2*).*

The maximal ideal of the vertex

$$\mathfrak{m} = (z_0, z_1, z_2)$$

is a homogeneous Poisson ideal. We denote $\mathcal{P}_n := \mathcal{P}/\mathfrak{m}^{n+1}$*.*

Theorem 1. *There exists an isomorphism of graded polynomial Poisson algebras*

$$\mathcal{P}_n \to \mathcal{P}(\mathcal{O}_n),$$

$$(z_0, z_1, z_2) \mapsto (y_0, y_1, y_2).$$

Proof. By Lemma 4 we know that the algebra $\mathcal{P}(\mathcal{O}_n)$ is generated by y_p's for $p = 0, ..., 2n$. By (9), (10) from the proof of Lemma 7 we have

$$\frac{1}{(2l)!}\mathrm{ad}_x^{2l}(\delta_2^l) = \delta_0^l = \frac{1}{(2l)!}\mathrm{ad}_x^{2l}(\delta_{2l}),$$

$$\frac{1}{(2l-1)!}\mathrm{ad}_x^{2l-1}(\delta_1\delta_2^{l-1}) = \delta_0^l = \frac{1}{(2l-1)!}\mathrm{ad}_x^{2l-1}(\delta_{2l-1}),$$

which means that in $\mathcal{P}(\mathcal{O}_n)$

$$y_{2l} = y_2^l, \quad y_{2l-1} = y_1y_2^{l-1}.$$

Therefore $\mathcal{P}(\mathcal{O}_n)$ is generated by (y_0, y_1, y_2). By the multiplicative law (1) from Lemma 3 we have

$$\frac{1}{2}\mathrm{ad}_x^2(\delta_1^2) = (\mathrm{ad}_x(\delta_1))^2 = \delta_0^2,$$

$$\frac{1}{2}\mathrm{ad}_x^2(\delta_0\delta_2) = \delta_0\frac{1}{2}\mathrm{ad}_x^2(\delta_2) = \delta_0^2.$$

Subtracting these equations we get

$$\frac{1}{2}\mathrm{ad}_x^2(\delta_1^2 - \delta_0\delta_2) = 0,$$

which implies that in $\mathcal{P}(\mathcal{O}_n)$

$$y_1^2 - y_0 y_2 = 0. \tag{11}$$

This relation is also an immediate consequence of the distinguished homomorphism.

Now using Lemma 4 we can write down the basis of $\mathcal{P}^p(\mathcal{O}_n)$ for every $p = 0, ..., 2n$

$$y_2^l, y_0 y_2^l, ..., y_0^{n-l} y_2^l, \qquad \text{for } \ p = 2l, \tag{12}$$

$$y_1 y_2^l, y_0 y_1 y_2^l, ..., y_0^{n-l-1} y_1 y_2^l, \qquad \text{for } \ p = 2l+1. \tag{13}$$

Since $\delta_0^{n+1} = 0$ we have

$$\frac{1}{p!}\mathrm{ad}_x^p(\delta_0^{n+1-v_p}\delta_p) = \delta_0^{n+1-v_p}\delta_0^{v_p} = \delta_0^{n+1} = 0,$$

which implies that

$$y_0^{n+1-v_p} y_p = 0.$$

By Lemma 7 this is equivalent to

$$y_0^{n+1-l} y_{2l-1} = 0, \quad y_0^{n+1-l} y_{2l} = 0,$$

and the latter by (11) is equivalent to

$$y_0^{n+1-l} y_1 y_2^{l-1} = 0, \quad y_0^{n+1-l} y_2^l = 0.$$

Again by (11) this implies that for all possible p_0, p_1, p_2 such that $p_0 + p_1 + p_2 = n+1$

$$y_0^{p_0} y_1^{p_1} y_2^{p_2} = 0. \tag{14}$$

Looking at the basis (12)-(13) we see that (11) and (14) form together a complete system of relations on generators y_0, y_1, y_2 which proves that the application of graded algebras $\mathcal{P}_n \to \mathcal{P}(\mathcal{O}_n)$, $z_p \mapsto y_p$ for $p = 0, 1, 2$ is well defined and is an isomorphism. The equality $\mathrm{ad}_x(\delta_1) = \delta_0$ implies $\{y_0, y_1\} = y_0$ and by the construction of $\delta_1 := 1/2\ \mathrm{ad}_x(\delta_2)$ we have $\{y_0, y_2\} = 2y_1$. The following calculation

$$\begin{aligned}\mathrm{ad}_x([\delta_1, \delta_2] - \delta_2) &= [\mathrm{ad}_x(\delta_1), \delta_2] + [\delta_1, \mathrm{ad}_x(\delta_2)] - \mathrm{ad}_x(\delta_2) = \\ &= [\delta_0, \delta_2] + [\delta_1, 2\delta_1] - [\delta_0, \delta_2] = 0\end{aligned}$$

shows that $\{y_1, y_2\} = y_2$. This relation is also an immediate consequence of the distinguished choice of $\delta_0, \delta_1, \delta_2$. Therefore the above isomorphism of graded algebras preserves the Poisson structure. □

Corollary 5. $\mathrm{Spec}(\mathcal{P}(\mathcal{O}_n))$ *admits a structure of a conical Poisson scheme with a hamiltonian infinitesimal algebraic action of SL_2, whose moment map is a closed embedding onto the n-th infinitesimal neighborhood of the vertex in the nilpotent cone in* sl_2^*.

Theorem 2. *The distinguished homomorphism of filtered algebras*

$$\mathcal{U}(\mathrm{sl}_2)/(C - n(n+2)) \to \mathcal{D}(\mathcal{O}_n),$$

is surjective with kernel (e^{n+1}).

Proof. Since the principal symbols y_0, y_1, y_2 of the distinguished operators $\delta_0, \delta_1, \delta_2$ generate the associated graded algebra $\mathcal{P}(\mathcal{O}_n)$ of the filtered algebra $\mathcal{D}(\mathcal{O}_n)$ then the distinguished operators generate $\mathcal{D}(\mathcal{O}_n)$ as well, which proves the surjectivity of the distinguished homomorphism. Since all maximal ideals in $\mathcal{U}(\mathrm{sl}_2)$ of height two are of the form $(e^{n+1}, C - n(n+2))$, $n \geq 0$ ([13], Theorem 4.5 (ii)) this proves our theorem. □

4. Principal symbols and inverse limits. Note that in general the graded Poisson algebra spanned by principal symbols is not functorial with respect to a given commutative algebra. However graded Poisson algebras spanned by principal symbols of the surjective inverse system $\mathcal{O}_{n+1} \to \mathcal{O}_n$ (when applied objectwise) form a surjective inverse system $\mathcal{P}(\mathcal{O}_{n+1}) \to \mathcal{P}(\mathcal{O}_n)$ as well, and we have

$$\lim_n \mathcal{P}(\mathcal{O}_n) = k[[y_0, y_1, y_2]]/(y_1^2 - y_0 y_2). \tag{15}$$

On the other hand, for $\mathcal{O} = k[x]$ we have $\mathcal{P}(\mathcal{O}) = k[x_0, x_1]$, where x_0 and x_1 are principal symbols of operators x and $\frac{\mathrm{d}}{\mathrm{d}x}$, respectively. Fix a maximal ideal $\mathfrak{m} = (x_0, x_1) \subset \mathcal{P}(\mathcal{O})$ and denote $\mathcal{P}(\mathcal{O})_n := \mathcal{P}(\mathcal{O})/\mathfrak{m}^n$. Since

$$\{x_0, x_1\} = -1 \tag{16}$$

this ideal and all its powers are not Poisson, so $\mathcal{P}(\mathcal{O})_n$ are not Poisson algebras. However

$$\{\mathfrak{m}^m, \mathfrak{m}^n\} \subset \mathfrak{m}^{m+n-1}, \tag{17}$$

hence the inverse limit

$$\lim_n \mathcal{P}(\mathcal{O})_n = k[[x_0, x_1]] \tag{18}$$

inherits a graded Poisson structure from $\mathcal{P}(\mathcal{O})$.

It is an interesting fact that these two complete graded Poisson $\lim_n \mathcal{O}_n$-algebras can be compared in a unique way. More presisely, we have the following theorem.

Theorem 3. *There exists the unique grading preserving Poisson homomorphism of graded Poisson algebras making the following diagram commutative*

$$\begin{tikzcd} \lim_n \mathcal{P}(\mathcal{O})_n & & \lim_n \mathcal{P}(\mathcal{O}_n) \arrow[ll, "\exists!"'] \\ & \lim_n \mathcal{O}_n \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Proof: We will look for a solution of the above problem using the explicit presentation

$$\begin{tikzcd} k[[x_0, x_1]] & & k[[y_0, y_1, y_2]]/(y_1^2 - y_0 y_2) \arrow[ll, "\exists?"'] \\ & k[[x]] \arrow[ul] \arrow[ur] & \end{tikzcd},$$

where the structure of $k[[x]]$-algebra is given by $x \mapsto y_0$ and $x \mapsto x_0$, respectively.

As a homomorphism of $k[[x]]$-algebras the map $k[[y_0, y_1, y_2]]/(y_1^2 - y_0 y_2) \to k[[x_0, x_1]]$ should be of the form

$$y_0 \mapsto x_0, \quad y_1 \mapsto f_1, \quad y_2 \mapsto f_2,$$

for some $f_1, f_2 \in k[[x_0, x_1]]$ such that

$$f_1^2 = x_0 f_2, \tag{19}$$

which implies that

$$f_1 = x_0 g, \quad f_2 = x_0 g^2, \tag{20}$$

for some $g \in k[[x_0, x_1]]$.

This is grading preserving if and only if g is of degree one, hence of the form

$$g = h x_1, \tag{21}$$

for some h of degree zero, which means that $h \in k[[x_0]] \subset k[[x_0, x_1]]$. Consequently

$$f_1(x_0, x_1) = x_0 h(x_0) x_1, \quad f_2(x_0, x_1) = x_0 h(x_0)^2 x_1^2. \tag{22}$$

This is a Poisson homomorphism if and only if

$$\{x_0, f_1\} = x_0, \quad \{x_0, f_2\} = 2f_1, \quad \{f_1, f_2\} = f_2, \tag{23}$$

which by (16) and (22) is equivalent to

$$h = -1. \tag{24}$$

All this means that there exists the unique solution to our problem and it is of the form

$$y_0 \mapsto x_0, \quad y_1 \mapsto -x_0 x_1, \quad y_2 \mapsto x_0 x_1^2. \quad \square \tag{25}$$

It is a very interesting homomorphism. First of all, it is the completion of a (birational) resolution of singularity in the category of graded Poisson algebras. Geometrically, this resolution describes the following diagram in the category of conical Poisson varieties

$$\begin{tikzcd} \mathrm{T}^*\mathbb{A}^1 \arrow[rr, "\exists!"] \arrow[dr] & & Y \arrow[dl] \\ & \mathbb{A}^1 & \end{tikzcd},$$

where $Y = \operatorname{Spec}(k[y_0, y_1, y_2]/(y_1^2 - y_0 y_2))$ is a quadratic cone. Left hand side morphism is a family of lines which is mapped into a family of conics degenerating to a double line $(y_0 = 0, y_1^2 = 0)$ on the right hand side. This resolution contracts the line $(x_0 = 0)$ to the vertex of the cone Y and the scheme theoretical pre-image of the double line is the line $(x_0 = 0)$ again.

5. Infinitesimal-birational duality. Another remarkable property of this morphism deserves a separate theorem. The following theorem means that the above canonical resolution of singularity together with the Fourier transformation establish a **duality** between the above SL_2-symmetries of principal symbols of the system of infinitesimal neighborhoods of finite order of a point in the line $\mathbb{A}^1$ (**infinitesimal** picture) and automorphisms of dual principal symbols induced by birational automorphisms of the dual line $\hat{\mathbb{A}}^1$ (**birational** picture).

Theorem 4. "Magic Diamond Theorem". *The following diagram of canonical morphisms in the category of conical Poisson varieties*

$$\begin{tikzcd}
& \mathrm{T}^*\mathbb{A}^1 \arrow[dl, "\text{resol. sing.}"'] \arrow[dr, "\text{cl. Fourier tr.}"] & \\
Y \arrow[dr, "\text{inf. moment m.}"'] & & \mathrm{T}^*\hat{\mathbb{A}}^1 \arrow[dl, "\text{bir. moment m.}"] \\
& \mathrm{sl}_2^* &
\end{tikzcd}$$

is commutative.

Proof: The left hand side of this diagram is already described in Corollary 5 and Theorem 3. The composition on the left hand side reads as

$$k[z_0, z_1, z_2] \to k[x_0, x_1],$$

$$z_0 \mapsto x_0, \quad z_1 \mapsto -x_0x_1, \quad z_2 \mapsto x_0x_1^2. \tag{26}$$

The classical Fourier transformation on the right hand side reads as

$$\mathrm{T}^*\hat{\mathbb{A}}^1 \leftarrow \mathrm{T}^*\mathbb{A}^1,$$

$$k[\hat{x}_0, \hat{x}_1] \to k[x_0, x_1],$$

$$\hat{x}_0 \mapsto x_1, \quad \hat{x}_1 \mapsto -x_0. \tag{27}$$

The infinitesimal form of the rational action of the algebraic group SL_2 on $\mathrm{T}^*\hat{\mathbb{A}}^1$ canonically induced from the canonical action of SL_2 on $\hat{\mathbb{A}}^1$ by birational automorphisms

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (\hat{x}_0, \hat{x}_1) = \left(\frac{a\hat{x}_0 + b}{c\hat{x}_0 + d}, (c\hat{x}_0 + d)^2\hat{x}_1\right)$$

reads as follows

$$e \mapsto -\frac{\partial}{\partial \hat{x}_0} = \{-\hat{x}_1, -\}, \tag{28}$$

$$h \mapsto -2\hat{x}_0\frac{\partial}{\partial \hat{x}_0} + 2\hat{x}_1\frac{\partial}{\partial \hat{x}_1} = \{-2\hat{x}_0\hat{x}_1, -\}, \tag{29}$$

$$f \mapsto \hat{x}_0^2\frac{\partial}{\partial \hat{x}_0} - 2\hat{x}_0\hat{x}_1\frac{\partial}{\partial \hat{x}_1} = \{\hat{x}_0^2\hat{x}_1, -\}. \tag{30}$$

Therefore the moment map associated with this action on $\mathrm{T}^*\hat{\mathbb{A}}^1$ is of the form

$$k[z_0, z_1, z_2] \to k[\hat{x}_0, \hat{x}_1],$$

$$z_0 \mapsto -\hat{x}_1, \quad z_1 \mapsto -\hat{x}_0\hat{x}_1, \quad z_2 \mapsto -\hat{x}_0^2\hat{x}_1.$$

It is easy to see that its composition with the classical Fourier transformation (27) gives the same as (26), what proves the commutativity of the above square. □

Remark. The moment map on $\mathrm{T}^*\hat{\mathbb{A}}^1$ is the restriction of the canonical moment map on $\mathrm{T}^*\hat{\mathbb{P}}^1$ associated with the action of the automorphism group of $\hat{\mathbb{P}}^1$. One has the following commutative diagram

$$\begin{tikzcd} \mathrm{T}^*\hat{\mathbb{P}}^1 \arrow[d, "\text{moment m.}"'] & \tilde{\mathcal{N}} \arrow[l, "\cong"'] \arrow[d, "\text{Springer res.}"] \\ \mathrm{sl}_2^* & \mathcal{N} \arrow[l, "\text{Killing f.}"'] \end{tikzcd}$$

where the right-hand vertical morphism is the Springer resolution of the nilpotent cone $\mathcal{N} \subset \mathrm{sl}_2$ [26]. Therefore, by Theorem 4, our canonical resolution as in Theorem 3 is, up to the classical Fourier transformation and the isomorphism provided by the Killing form, essentially nothing but the Springer resolution.

6. Quantization of the infinitesimal-birational duality. The following theorem shows and explains the quantum nature of the above duality.

Theorem 5. "Quantized Magic Diamond Theorem". *There exists the following commutative diagram in the category of almost commutative algebras*

$$\begin{tikzcd} & \mathcal{D}(\mathbb{A}^1) & \\ \mathcal{U}(\mathrm{sl}_2)/(C - n(n+2)) \arrow[ur, "\text{q. resol. sing.}"] & & \mathcal{D}(\hat{\mathbb{A}}^1) \arrow[ul, "\text{q. Fourier tr.}"'] \\ & \mathcal{U}(\mathrm{sl}_2) \arrow[ul, "\text{q. inf. moment m.}"] \arrow[ur, "\text{q. bir. moment m.}"'] & \end{tikzcd}$$

which is a quantization of the Magic Diamond.

Proof: The only nonobvious arrows in this diagram are: the quantized birational moment map on the right hand side and the quantized resolution of singularities on the left hand side. The first one is canonically obtained from the birational action of SL_2 on $\hat{\mathbb{A}}^1$ as above, i.e.

$$e \mapsto -\frac{d}{d\hat{x}}, \quad h \mapsto -2\hat{x}\frac{d}{d\hat{x}}, \quad f \mapsto \hat{x}^2\frac{d}{d\hat{x}}. \tag{31}$$

Composing with the quantum Fourier transformation we get

$$e \mapsto x, \quad h \mapsto 2x\frac{d}{dx} + 2, \quad f \mapsto -x^2\frac{d^2}{dx^2} - 2\frac{d}{dx}. \tag{32}$$

Note that these operators look exactly like the operators (8) inducing the distinguished differential operators on the n-th infinitesimal neighborhood of a point in $\mathbb{A}^1$, except the strange value $n = -2$. Nevertheless, they satisfy relation $h^2 + 2(ef + fe) = n(n+2)$ for $n = -2$, which means that the composite homomorphism factorizes as in the left hand side of the diagram. This proves the commutativity of this diagram. Passing to the diagram of associated graded Poisson algebras is equivalent to replacing all differential operators (32) by their principal symbols and all differential operators (31) by their classes modulo differential operators with polynomial coefficients of lower degree. It is easy to see that we reobtain the Magic Diamond. $\square$

Remark. The strange value $n = -2$ in the above proof justifies our definition of quantization as Morita equivalence class. Namely, for $n \geq 0$ the primitive quotient $\mathcal{U}(\mathrm{sl}_2)/(C - n(n+2))$ is a representative of a quantization of the nilpotent cone admitting closed embedding (surjective homomorphism of algebras) of a representative of a quantization of n-th infinitesimal neighborhood of the vertex (see Corollary 5 and Theorem 2). But by the Beilinson-Bernstein Theorem these primitive quotients are Morita equivalent for all $n \in \mathbb{Z}$. This means that this quantization can be computed as the class of any primitive quotient $\mathcal{U}(\mathrm{sl}_2)/(C - n(n+2))$, even for $n = -2$ which has no any immediate geometric meaning.

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